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FUNDAMENTAL STATEMENTS OF THE PHENOMENOLOGICAL APPROACH IN NON-LOCAL HYDRODYNAMICS†

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A system of postulates of non-local macroscopic phenomological hydrodynamics is proposed based on the results of the microscopic theory previously obtained. Space time non-locality holds in the material relations of the theory. The implementation of an axiomatic scheme is presented for the case of a multicomponent mixture with viscosity and thermal conduction. In this case, the classical transport laws are obtained in the limit of long waves and slow processes. It is proved that, within the framework of non-local hydrodynamics, it is possible to combine the dissipation and the finite velocity of propagation of a signal in a consistent manner. A model functional form of the kernels is proposed which satisfies all the requirements of the theory. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The Chapman–Enskog method [1–3] enables one, when there is a certain small parameter in the kinetic equation (that is, in the equation for the one particle distribution function of the molecules of the medium), to obtain a system of hydrodynamic equations and to derive, in the second order of perturbation theory, the classical transport laws of Newton, Fourier and Fick. The equations of an ideal medium are obtained in the first order of perturbation theory. Generalizations and modifications of the Chapman–Enskog theory have been developed which are based on various assumptions regarding the occurrence of small and large parameters in the kinetic equation as well as on various asymptotic expansion schemes [4–6]. However, these methods do not work if there is no corresponding small or large parameter. Furthermore, the question remains open as to whether a particular solution of the kinetic problem corresponds to an arbitrary solution of the hydrodynamic problem.

It has recently been shown that, for a specified form of the sources or external forces, not only kinetic theory [7–10] but also quantum and classical statistical mechanics are exactly equivalent to non-local hydrodynamics [11–17]. A space-time non-locality arises in the material relations, that is, in the expressions for the flows of the hydrodynamic quantities. This equivalence means that, in the case of an arbitrary solution of the hydrodynamic problem, it is possible to reconstruct the corresponding process in kinetics and in quantum and classical statistics. The relation of these last results to the Chapman–Enskog results is the same as that between any exact theory and an asymptotic method: it is possible to use the Chapman–Enskog asymptotic forms in non-local hydrodynamics and to obtain the classical local transport laws.

Other approaches to the derivation of the equations of non-local hydrodynamics from the kinetic equation or from the classical or quantum Liouville equation have been implemented [18–20]. In this case, the initial dynamical equations are taken in a form without external forces or sources. As was shown in [17], this fact does not enable one to calculate the non-local material relations in a unique manner. The authors were therefore forced to eliminate possible arbitrariness implicitly in a "by hands" manner. The issue of the realizability of the inverse transition from, a hydrodynamic description to a kinetic or statistical description was also not touched upon.

In this paper, the fundamental statements of non-local hydrodynamics, which are required to construct models of continua and to solve actual problems in mechanics are systematized without recourse to microscopic dynamical theories.

Since the non-locality which is being considered covers the case of temporal non-locality, that is, heredity, the proposed theory has certain common features with rational mechanics [21–23], which describes materials with complex rheology and heat conduction which are determined by the past history of the deformations of a particle of the medium. Rational mechanics has the structure of an axiomatic

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theory without a connection with the laws of physics; the correctness of the theory must therefore be verified experimentally for each actual material. In this respect, the following fundamental differences exist between the theory proposed and the approach in [21–23]: (a) a spatial non-locality is included in the treatment, (b) the theory encompasses all phenomena and processes in continua and not only rheology and heat conduction, (c) the fundamental statements of non-local hydrodynamics are rigorously proved in fundamental physics, that is, beyond the limits of what is properly classical mechanics.

Apart from rational mechanics, another phenomenological theory, which enables on to take account of relaxation processes in a particle of a medium, known as extended thermodynamics [24], is being developed. The fundamental idea behind this theory lies in the extension of the set of characteristics of a particle of a medium for which a hierarchy of relaxation differential equations is written out. Extended thermodynamics is formally and mathematically embedded in non-local hydrodynamics: it corresponds to the special case when the Fourier images of the relaxation kernels in the material relations have a discrete set of poles in the complex frequency plane. The opposite is not true, and non-local hydrodynamic models with a continuous spectrum of relaxation times cannot be described within the framework of extended thermodynamics.

Note that, in principle, non-local hydrodynamics can be constructed solely on the basis of the internal laws of mechanics [25–27], as is the case for extended thermodynamics. However, significantly weaker results are obtained when this is done.

A non-relativistic approximation is used below and, for simplicity, we do not touch upon questions associated with boundary conditions: it is assumed that the medium occupies the whole space and that a state of rest or equilibrium translational motion is realized at infinity. The Greek superscripts take the values 0, 1, 2, 3, corresponding to a certain inertial measurement system x^α, where x^0 is the time. The Latin superscripts a, b, c, d take the values 1, 2, 3, corresponding to the spatial Cartesian coordinates. Summation is carried out over repeated indices.

2. FUNDAMENTAL PROPOSITIONS

In hydrodynamic models, the complete system of governing equations has the form of local conservation laws for certain mechanical quantities: energy, momentum and so on. Suppose that $\overset{0}{Q}_A$ is a complete set of densities of such quantities, which are assumed to be functionally independent. Henceforth, the Latin subscripts A, B and C take the values $0, \ldots, (N-1)$, where N is the number of independent conservation laws. At a fixed instant of time, there is a set of distributions $\overset{0}{Q}_A = \overset{0}{Q}_A(x^a)$ which, at infinity, tend to the constant values $\overset{0}{Q}_{A0}$, which correspond to the equilibrium state of the medium. We shall assume that deviations from equilibrium $\Delta\overset{0}{Q}_A = \overset{0}{Q}_A - \overset{0}{Q}_A$ are Lebesgue summable.

Definition 1. We shall call the set of distributions $\overset{0}{Q}_A = \overset{0}{Q}_A(x^a)$ the instantaneous state of the medium. The norm in the space L^1 of the Lebesgue integrable functions $\Delta\overset{0}{Q}_A$ enables one to interpret the space of instantaneous states as an affine normed space.

The axioms of non-local hydrodynamics are then numbered as $A0, A1, A2, \ldots$.

$A0$. A smooth closed form

$$\Omega = \int F^A \delta \overset{0}{Q}_A \tag{2.1}$$

is defined in the space of the instantaneous states of the medium.

Generally speaking, the distributions $F^A = F^A(x^a)$ which occur in formula (2.1) are functionals which depend on the instantaneous state of the medium. It is assumed that, at spatial infinity, the quantities F^A reduce to certain equilibrium values F^A_0.

Lemma 1. A smooth functional H, the differential of which is identical to Ω, is defined in the space of the instantaneous states of the medium.

Proof. For a given instantaneous state $\overset{0}{Q}_{A1}(x^a)$, we choose a smooth process P: $\overset{0}{Q}_A = \overset{0}{Q}_A(t, x^a)$, $0 \leqslant t \leqslant 1$ such that

$$\overset{0}{Q}_{A0} = \overset{0}{Q}_A(0, x^a),\ \overset{0}{Q}_{A1}(x^a) = \overset{0}{Q}_A(1, x^a)$$

and we put

$$\delta\overset{0}{Q}_A = \frac{\partial}{\partial t}\overset{0}{Q}_A(t, x^a)dt,\ H = H[\overset{0}{Q}_{A1}(x^a)] = \int_P \Omega$$

The fact that H is independent of the choice the process follows from the fact that the form is closed.

Remark 1. The quantity H with the opposite sign is interpreted as the entropy of the system apart from a constant term.

Definition 2. The second order form $\Omega_1 = D^2H$ is the second differential of the functional H [28].
*A*1. In the case of an equilibrium instantaneous state, Ω_1 is a positive definite form.
The following lemma arises from *A*1 and the implicit function theorem.

Lemma 2. In a certain neighbourhood of the equilibrium state, the distributions $Q^0_A(x^a)$ can be expressed as functionals of the distributions $F^A(x^a)$.

Using the notation $g^0_A = Q^0_A - Q^0_{A0}, f^A = F^A - F^A_0$, we write out the corresponding dependence in the form of a Taylor series

$$g^0_A(x^b) = \int \lambda^a_{AB}(x^b - x^b_1) f^B(x^b_1) dx^b_1 + \\ + \int \lambda^a_{ABC}(x^b - x^b_1, x^b - x^b_2) f^B(x^b_1) f^C(x^b_2) dx^b_1 dx^b_2 + \ldots \tag{2.2}$$

We now define the matrix

$$\Lambda_{AB}(k_a) = \int \exp(-ik_a x^a) \lambda_{AB}(x^a) dx^a$$

According to *A*1, this matrix is Hermitian and positive definite

$$\Lambda = \Lambda^+ > 0 \tag{2.3}$$

*A*2. The dynamics of the medium is determined by the system of equations

$$\partial_\alpha Q^\alpha_A = s_A (\partial_\alpha = \partial / \partial x^\alpha) \tag{2.4}$$

where the hydrodynamic flows $Q^a_A = Q^a_A(x^\alpha)$ depend on the past history of the medium

$$Q^b_A(x^a) = Q^b_A[x^a, Q^0_B(x^0_* \leqslant x^0, x^a_*)] \tag{2.5}$$

and the sources $s_A = s_A(x^\alpha)$ describe the action of external systems on the medium.

It is assumed that, at spatial infinity, the flows tend to the equilibrium values Q^a_{A0}. Since, in the case of equilibrium states, the set of equilibrium densities Q^a_{A0} is decisive, the quantities Q^a_{A0} are functions of Q^a_{A0}.

Suppose a certain smooth process P: $Q^0_A = Q^0_A(x^\alpha), -\infty \leqslant x^0 \leqslant +\infty$, which starts and terminates with an equilibrium state, is obtained as a result of the action of sources (2.4). Then, by (2.4), the following equality holds

$$0 = \int_P \Omega = \Delta_1 + \Delta_2$$

$$\Delta_1 = \int F^A s_A dx^\alpha, \quad \Delta_2 = \int \partial_a F^A Q^a_A dx^\alpha$$

The following *a priori* constraint is imposed on the form of functional (2.4).
*A*3. The inequality $\Delta_2 \leqslant 0$ holds.

Traditionally, functional (2.5) is subdivided into an equilibrium component $Z^a{}_A$ and a non-equilibrium $D^a{}_A$ component

$$Q^a_A(x^\alpha) = Z^a_A(x^a) + D^a_A(x^a) \tag{2.6}$$

$$Z^b_A(x^a) = Z^b_A[x^a, Q^0_B(x^0, x^a_*)] \tag{2.7}$$

$$D^b_A(x^\alpha) = D^b_A[x^a, Q^0_B(x^0_x \leqslant x^0, x^a_*)] \tag{2.8}$$

although, of course, the introduction of these components in the case of the given functional (2.5) does not change the dynamical equations.

By definition, representation (2.6) must satisfy three conditions:

(a) for a process which starts out from and terminates at the same equilibrium state, the following relation holds

$$\int \partial_a F^A Z_A^a dx^\alpha = 0 \tag{2.9}$$

(b) if the equality

$$c^A Q_A^0 + c_a^A Q_A^a = 0 \tag{2.10}$$

is satisfied in the case of certain constants c^A, c_a^A, then the following equality holds

$$c_a^A D_A^a = 0 \tag{2.11}$$

(c) the component (2.8) identically vanishes for spatially constant Q_A^0.

Nevertheless, the assumptions concerning the representation (2.6) which have been listed does not enable one, in the case of a given functional (2.5), to determine the equilibrium flows Z_A^a uniquely without recourse to microscopic theory. It is simple to determine the flows Z_A^a in the special case when an additional condition is satisfied.

B. The quantities F^A, Z_A^a are determined by the values of the quantities Q_A^0 at the same point.

In the latter case, the dependence of the quantities F^A, Z_A^a on the values of Q_A^0 is the same as the dependence in the class of equilibrium states (that is, it is determined by ordinary equilibrium thermodynamics).

Remark 2. Condition *B* is not an axiom. It can be violated in quantum and classical statistical mechanics. It is always satisfied in kinetic theory.

Using Lemma 2, we express the flows (2.7) and (2.8) in the form of functional series in powers of f^A

$$\begin{aligned} Z_A^a(x^\alpha) &= Q_{A0}^a + \int z_{AB}^a(x^b - x_1^b) f^B(x^0, x_1^b) dx_1^b \\ &+ \int z_{ABC}^a(x^b - x_1^b, x^b - x_2^b) \times \\ &\times f^B(x^0, x_1^b) f^B(x^0, x_2^b) dx_1^b dx_2^b + \ldots \end{aligned} \tag{2.12}$$

$$\begin{aligned} D_A^a(x^\alpha) &= \int d_{AB}^{ac}(x^a - x_1^a) \partial_c f^B(x_1^\alpha) dx_1^\alpha + \\ &+ \int d_{ABC}^{ac}(x^\alpha - x_1^\alpha, x^\alpha - x_2^\alpha) \times \\ &\times \partial_c f^B(x_1^\alpha) f^B(x_2^\alpha) dx_1^\alpha dx_2^\alpha + \ldots \end{aligned} \tag{2.13}$$

and define the matrices

$$Z_{AB}^a(k_b) = \int \exp(-ik_b x^b) z_{AB}^a(x^b) dx^b$$

$$D_{AB}^{ab}(k_\alpha) = \int \exp(-ik_\alpha x^\alpha) d_{AB}^{ab}(x^\alpha) dx^\alpha$$

$$B_{AB}(k_\alpha) = -k_a k_b D_{AB}^{ab}(k_\alpha)$$

Note that the kernels $d_{AB}^{ac}(x_1^\alpha)$, $d_{ABC}^{ac}(x_1^\alpha, x_2^\alpha)$, . . . vanish when $x_n^0 < 0$ (causality) and hence, by the Paley–Wiener theorem [29], the functions $D_{AB}^{ab}(k_\alpha)$, $B_{AB}(k_\alpha)$ are holomorphic in the half plane $\operatorname{Im} k_0 \leqslant 0$.

Lemma 2. The following matrix relations hold

$$Z^a = Z^{a+} \tag{2.14}$$

$$B^+ + B \geq 0 \tag{2.15}$$

Proof. Substituting expressions (2.12) into (2.9), changing to the Fourier images, discarding terms of the second order with respect to the functions $f^A(x^\alpha)$ and using the arbitrariness in the choice of these functions, we obtain equality (2.14). Then, using *A*3 (2.6), (2.9) and (2.13) and the arbitrariness in the choice of the functions $f^A(x^\alpha)$, we derive inequality (2.15).

The matrix inequality (2.15) is the usual dissipation condition which arises in models with heredity [25]. However, microscopic theories give a stronger condition.

*A*4. The inequality

$$C_a^{A*}C_b^{B}\left(D_{AB}^{ab}+D_{BA}^{ba*}\right)\leqslant 0 \tag{2.16}$$

is satisfied in the case of the arbitrary complex quantities C_a^A.

Additional constraints on the kernels (an analogue of the Onsager conditions) follow from microreversibility. We shall assume that the equilibrium state at infinity is a state of rest. Suppose that, accompanying time reversal, the quantities Q_A^0 are multiplied by the numbers ε_A, $\varepsilon_A = \pm 1$.

*A*5. The following equalities hold

$$\begin{aligned}\Lambda_{AB}(k_c) &= \varepsilon_A\varepsilon_B\Lambda_{BA}(-k_c)\\ Z_{AB}^{a}(k_c) &= -\varepsilon_A\varepsilon_B Z_{BA}^{a}(-k_c)\\ D_{AB}^{ab}(k_0,k_c) &= \varepsilon_A\varepsilon_B D_{BA}^{ba}(k_0,-k_c)\end{aligned} \tag{2.17}$$

Remark 3. The axiom *A*5 imposes constraints on the kernels in linear perturbation theory. It follows from general considerations [30] that microreversibility implies constraints at all orders of perturbation theory. However, since the non-linear part of functionals (2.2), (2.12) and (2.13) have not been sufficiently well studied in microscopic theory at the present time, we shall confine ourselves here to a linear formulation.

3. HYDRODYNAMICS OF A MULTICOMPONENT MIXTURE

We shall now apply the mathematical scheme of Section 2 to the case of a multicomponent mixture without chemical reactions. Suppose M is the number of components. We shall assume that i and j take the values $1, \ldots, M$, corresponding to the number of components, and that I and J take the values 4, $5, \ldots, (M+3)$. If i and j and I and J are used in the same formula, their values are related by the equalities $I = i + 3$, $J = j + 3$. A, B and C take the values $0, 1, \ldots, (M+3)$, corresponding to the laws of conservation of energy, of the components of the momentum and the numbers of molecules of the components in the mixture. We shall assume that a state of rest is realized at spatial infinity and that, together with axioms *A*0–*A*4, condition *B* is satisfied.

Hence, Q_0^0 is the energy density, Q_a^0 is the density of the ath component of the momentum and $n_i = Q_I^0$ is the molar density of the component of the mixture with number I.

The parameters F^A in formula (2.1) allow of the following physical interpretation

$$F^0 = -\beta,\ F^a = \beta u^a,\ F^I = \beta\mu_i \tag{3.1}$$

where β is the inverse temperature, u^a are the components of the velocity of the medium, $\mu_i = (\mu_{i0} - 1/2m_iu^au^a)$, μ_{i0} is the chemical potential of the ith component of the stationary medium and m_i is the molar mass of the ith component.

Now, in order to give a more traditional form to the hydrodynamic equations (2.6), it is necessary to express the hydrodynamic 4-flows Q_A^α in terms of other quantities. We now introduce into consideration the mass density $\rho = \Sigma_i m_in_i$, the mean mass velocity with components $u^a = \rho^{-1}Q_a^0$, the diffusion fluxes with the components $I_i^a = (Q_I^a - n_iu^a)$, the kinetic energy density of the medium $K = 1/2\rho u^au^a$, the internal energy density of the medium $U = Q_0^0 - 1/2\rho u^au^a$, the stress tensor with the components $p^{ab} = (\rho u^au^b - Q_b^a)$ (which is assumed to be symmetric) and the thermal flux vector with the components $p^{ab} = (\rho u^au^b - Q_b^a - (U + K)u^a$. We emphasize that the new definition of the velocity of the medium with components u^a matches the old definition (3.1) in the case of equilibrium states.

System (2.4) can now be rewritten as

$$\begin{aligned}&\partial_0(U+K)+\partial_a\left(q^a-p^{ab}u^b+(U+K)u^a\right)=s_0\\ &\partial_0(\rho u^a)+\partial_b(-p^{ab}+\rho u^au^b)=s_a\\ &\partial_0 n_i+\partial_a(I_i^a+n_iu^a)=s_I\end{aligned}$$

We will now discuss the definition of the viscous stress tensor. In the case of equilibrium states, the stress tensor reduces to the spherical tensor $p^{ab} = -p\delta^{ab}$. Hence, in the class of equilibrium states, the pressure p can be expressed as a function of the internal energy U and the densities n_i. Using this functional relation, we extend the definition of pressure to non-equilibrium states. The components of the viscous stress tensor are then calculated using the formula

$$\tau^{ab} = p^{ab} + p\delta^{ab}$$

After the above transformations, in order to specify the material relations, it is sufficient to give expressions for the dissipative flows q^a, τ^{ab}, I_i^a. The remaining terms in the expressions for the hydrodynamic flows Q_A^a are the redesignated equilibrium states Z_A^a and are calculated in the class of equilibrium distributions. From the results in Section 2, we obtain

$$q^a = D_0^a, \quad \tau^{ab} = -D_b^a, \quad I_i^a = D_I^a \tag{3.2}$$

Note the identity of the form (2.10)

$$\sum_{i=1}^{M} m_i Q_I^a - Q_a^0 = 0$$

from which the equality

$$\sum_{i=1}^{M} m_i D_I^a = 0 \tag{3.3}$$

of the form (2.11) follows.

We will now study in greater detail the algebraic structure of relations (3.2) in the linear approximation. For an arbitrary function $g = g(x^\alpha)$, we shall denote its Fourier image by the symbol $g_F(k_\alpha)$

$$g_F = g_F(k_\alpha) = \int \exp(-ik_\alpha x^\alpha) g(x^\alpha) dx^\alpha$$

Retaining the principal terms in relations (3.2) and changing to Fourier images, we obtain, taking (2.13) into account

$$q_F^a = ik_b D_{0B}^{ab} f_F^B \tag{3.4}$$

$$\tau_F^{ab} = -ik_c D_{bB}^{ac} f_F^B \tag{3.5}$$

$$I_{iF}^a = ik_b D_{IB}^{ab} f_F^B \tag{3.6}$$

The equality

$$\sum_{i-1}^{M} m_i D_{IB}^{ab} = 0 \tag{3.7}$$

is obtained from (3.3) in the linear approximation.

We now assume the invariance of the theory with respect to the group of rotation $O(3)$ which means that the medium is isotropic. We then have

$$(r(g)D)_{AB}^{ab}(k_0, \; gk_b) = D_{AB}^{ab}(k_0, \; k_b) \tag{3.8}$$

for any $g \in O(3)$. Here r are $O(3)$ groups which have been introduced in the linear space Φ of quantities of the form D_{AB}^{ab}. In Φ, we now pick out the maximum set of linearly independent invariants with respect to a subgroup which preserves the 3-vector k_b: $I^n = I_{AB}^{nab}(k_c)$. These invariants are chosen such that

$$(I_{AB}^{nab}(k_c))^* = I_{AB}^{nab}(-k_c) \tag{3.9}$$

The most general representation, which is compatible with (3.8), then has the form

$$D_{AB}^{ab} = I_{AB}^{nab} X_n \tag{3.10}$$

Here $X_n = X_n(k_\alpha)$ are scalar functions. In accordance with (3.9) and the reality of the Fourier prototypes d_{AB}^{ab}, these functions satisfy the condition

$$(X_n(k_\alpha))^* = X_n(-k_\alpha) \tag{3.11}$$

and they are therefore the Fourier images of certain real kernels $Y_n = Y_n(x^\alpha)$: $Y_{nF} = X_n$. Since the functions X_n are holomorphic with respect to the variable k_0 in the lower complex half-plane, then, by the Paley–Wiener theorem [29], it follows that the functions $Y_n(x^\alpha)$ vanish when $x^0 < 0$ (causality).

The algebraic structure the coefficients D^{ab}_{AB} enables us to write out an expansion of the form (3.10) which is invariant under the group of rotations. Here, the symmetry conditions $D^{ab}_{cA} = D^{cb}_{aA}$, $D^{ab}_{Ac} = D^{ac}_{Ab}$ (consequences of the symmetry of the stress tensor) and the reciprocity conditions (2.17) should be kept in mind. It is therefore sufficient to write out representation (3.10) for the following components

$$
\begin{aligned}
D^{ab}_{00} &= \delta_{ab}X_0 + k_a k_b X_1 \\
D^{ab}_{0c} &= ik_a\delta_{bc}X_2 + (ik_b\delta_{ac} + ik_c\delta_{ab})X_3 + ik_a k_b k_c X_4 \\
D^{ab}_{0I} &= \delta_{ab}X_{5I} + k_a k_b X_{6I} \\
D^{ab}_{cd} &= \delta_{ac}\delta_{bd}X_7 + (\delta_{ab}\delta_{cd} + \delta_{cb}\delta_{ad})X_8 + \delta_{ac}k_b k_d X_7 + \\
&+ (k_a k_b \delta_{cd} + k_c k_b \delta_{ad} + k_a k_d \delta_{cb} + k_c k_d \delta_{ab})X_8 + k_a k_b k_c k_d X_9 \\
D^{ab}_{cI} &= (ik_a\delta_{bc} + ik_c\delta_{ab})X_{10I} + ik_b\delta_{ac}X_{11I} + ik_a k_b k_c X_{12I} \\
D^{ab}_{IJ} &= \delta_{ab}X_{13IJ} + k_a k_b X_{14IJ}
\end{aligned}
$$

We now consider material relations (3.4)–(3.6) in the limit of small gradients, that is, in the linear approximation with respect to the wave number k_a

$$q^a_F = ik_a(\xi_0 f^0_F + \xi_{5I} f^I_F) \tag{3.12}$$

$$\tau^{ab}_F = -\delta_{ab}\xi_7 ik_d f^d_F - \xi_8(ik_a f^b_F + ik_b f^a_F) \tag{3.13}$$

$$d^a_{iF} = ik_a(\xi_{5I} f^0_F + \xi_{13IJ} f^J_F) \tag{3.14}$$

Here ξ_7, ξ_8, ξ_0, ξ_{5I}, ξ_{13IJ} are functions of the parameter k_0 which are identical to X_7, X_8, X_0, X_{5I}, X_{13IJ} when $k_a = 0$. Relations (3.12)–(3.14) are spatially local material relations of linear hydrodynamics with heredity. In the limit of slow processes ($k_0 \to 0$), the functions ξ_7, ξ_8, ξ_0, ξ_{5I}, ξ_{13IJ} tend to the usual transport coefficients for a multicomponent, viscous, heat-conducting gas.

On applying this procedure to equality (3.13), we obtain expressions for the bulk viscosity η_V and the shear viscosity η_S

$$\eta_V = -\beta\left(X_7 + \frac{2}{3}X_8\right)\Big|_{k_a=0}, \quad \eta_S = -\beta X_8\big|_{k_a=0}$$

By considering relation (3.12) when $f^I = 0$, we pick out a component which is proportional to the temperature gradient and the thermal conductivity $\varkappa$

$$\varkappa = -\beta^2 X_0\big|_{k_a=0}$$

Finally, from (3.14), it is possible to separate out the matrix of the coefficients which relates the diffusion fluxes and the gradients of the chemical potentials

$$D_{ij} = -\beta X_{13IJ}\big|_{k_a=0}$$

The fact that the transport coefficients are real follows from condition (3.11). Inequality (2.15) ensures the non-negativeness of the coefficients η_V, η_S, $\varkappa$ and of the matrix D_{ij}. The symmetry of the matrix D_{ij} follows from reciprocity conditions (2.16). In accordance with (3.7)

$$\sum_{i=1}^{M} m_i D_{ij} = 0$$

Hence, in the limit of slow processes and long waves, non-local hydrodynamics is reduced to hydrodynamics with classical transport laws.

4. SIGNAL PROPAGATION

A well-known deficiency of models with classical transport laws is the infinite velocity of signal propagation.

Starting from [31], attempts have been made to construct models of transport processes with a finite signal velocity. Generalizing these attempts, we can say that dissipation and a finite signal velocity can be successfully and consistently combined by introducing temporal non-locality in one form or another into the theory [32–34]. The case when, apart from the temporal non-locality, there is also a spatial non-locality, has been considered previously in [35].

We will show how dissipation and a finite signal velocity are combined within the framework of the axiomatics proposed in Section 2. Suppose υ is a positive constant with the dimension of velocity. We will describe a class of models in which the signal velocity does not exceed υ. Here, we shall use the theory in [36] which is based on the method of analytic expansion to complex values of the wave 4-vector k_α.

We shall assume that condition B is satisfied and also that the medium is stationary at spatial infinity.

Suppose s are small quantities, that is, the case of weak sources holds. According to (2.4), (2.2), (2.12) and (2.13) the propagation of perturbations in the Fourier images is described by the equation

$$Af_F = s_F, \quad A = ik_0\Lambda + ik_aZ^a + B \tag{4.1}$$

In particular, the dispersion equation for free waves has the form

$$\det A = 0 \tag{4.2}$$

We will use the following notation

(a) T_1 is the cone of the future in space-time

$$T_1 = \{x^0 \geqslant 0,\ (x^0)^2 \geqslant v^{-2}x^ax^a\}$$

(b) T_2 is a tube in the complex space of the wave 4-vectors $k_\alpha = \alpha_a + i\beta_\alpha$ (α_α, β_α are real)

$$T_2 = \{\beta_0 < 0,\ (\beta_0)^2 > v^2\beta_a\beta_a\}$$

We assume that the signal transmission velocity in the material relations does not exceed υ (Condition 1). In formal language, this means that the kernel $d^{ab}_{AB}(x^\alpha)$ vanishes outside the cone T_1. Then, according to the well-known results in [36], the matrix B is holomorphic in T_2, and it follows from this that the matrix A is also holomorphic in T_2.

Next, we assume that inequality (2.15) still holds in the tube T_2 (Condition 2).

Finally, we assume that the inequality

$$y^{A*}y^B\Lambda_{AB} \geqslant v^{-1}\,|y^{A*}y^BZ^1_{AB}| \tag{4.3}$$

is satisfied for any set of complex numbers y^A (Condition 3).

This assumption ensures that, when there is no dissipation (in the case of an ideal medium), the propagation velocity of perturbations does not exceed υ.

Suppose y^A is an arbitrary non-zero set of complex numbers. By Condition 2 and relations (2.3) and (2.4) the inequality

$$\mathrm{Re}(y^{A*}y^BA_{AB}) \geqslant -\beta_0y^{A*}y^B\Lambda_{AB} - \beta_ay^{A*}y^BZ^a_{AB}$$

is satisfied in the tube T_2.

It follows from this and from inequality (4.3) that, in the tube T_2

$$\mathrm{Re}(y^{A*}y^BA_{AB}) > 0$$

and hence Eq. (4.2) cannot be satisfied at any single point of T_2.

Suppose s_A is an instantaneous source of momentum

$$s_A = a_A\delta(x^\alpha)$$

a_A is a constant vector and $s_{AF} = a_A$. We now find the solution of Eq. (4.1)

$$f_F = A^{-1}a$$

It follows from the preceding results that this formula defines a vector function which is holomorphic in the tube T_2. Exponential singularities are possible on the boundary of the tube.
On carrying out an inverse Fourier transformation, we obtain the function $f_A(x^\alpha)$, which is equal to zero outside the cone T_1 as was required.

Hence, the additional Conditions 1–3 imposed on the hydrodynamic model lead to a state of affairs where the propagation velocity of small perturbations does not exceed υ.

5. FUNCTIONAL FORM OF THE KERNELS

When constructing non-local hydrodynamic models in a phenomenological approach, it is necessary to define model expressions for the kernels. For a sufficiently general approach, it is reasonable to use expressions of the form

$$X = -\int \varphi_0(\tau) d\tau (ik_0 + \varphi_1(\tau))((ik_0 + \varphi_1(\tau))^2 + k_a k_a \varphi_2(\tau))^{-1} \tag{5.1}$$

for the scalar functions X_n, where τ is a continuous parameter and $\varphi_0(\tau)$, $\varphi_1(\tau)$, $\varphi_2(\tau)$ are non-negative functions. Expression (5.1) satisfies condition (3.11), it is invariant under the group of rotations and is holomorphic when $\operatorname{Im} k_0 < 0$ (causality). Moreover, the inequality $\operatorname{Re} X < 0$ is satisfied, which enables one to satisfy requirement (2.16). Expression (5.1), depending on the choice of the functions $\varphi_0(\tau)$, $\varphi_1(\tau)$, $\varphi_2(\tau)$ (including the case of $\varphi_0(\tau)$ in the form of a sum of δ-functions), covers the majority of models and exact theoretical expressions for hydrodynamic kernels.

6. CONCLUSION

The axiomatic scheme of phenomenological hydrodynamics which has been proposed is intended for describing rapidly occurring processes or short waves in the bulk of a liquid or a gas. The theory can be generalized by taking account of the boundary conditions. Additional non-local terms, describing the effect of the boundaries, and then introduced into the material relations. Non-local hydrodynamics is reduced to classical hydrodynamics in the limit of slow processes and long waves and, therefore, non-local hydrodynamics formally encompasses all domains of application of classical hydrodynamics. However, non-local effects, in fact, really manifest themselves when classical hydrodynamics are no longer applicable, such as in problems of the propagation of ultrasound.

REFERENCES

1. CHAPMAN, S. and COWLING, T. G., *The Mathematical Theory of Non-uniform Gases. An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases.* Cambridge University Press, Cambridge, 1952.
2. FERZIGER, J. H. and KAPER, H. G., *Mathematical Theory of Transport Processes in Gases.* North-Holland, Amsterdam, 1972.
3. CERCIGNANI, C., *Theory and the Application of the Boltzmann Equation.* Scottish Academic Press, Edinburgh, 1975.
4. WALLANDER, S. V., NAGNIBEDA, Ye. A. and RYDALEVSKAYA, M. A., *Some Problems in the Kinetic Theory of a Chemically Reacting Mixture of Gases.* Izd. LGU, Leningrad, 1977.
5. ALEKSEYEV, B. V., Hydrodynamic equations in the kinetic theory of reacting gases. *Zh. Vychisl. Mat. Mat. Fiz.*, 1987, **27**, 730–740.
6. SHAPIRO, A. A., Kinetic theory of the seepage of a rarefied gas in an isotropic porous medium. *Teoret. Osnovy Khim. Tekhnologii*, 1993, **27**, 155–164.
7. DINARIYEV, O. Yu., Propagation of small perturbations in a relativistic gas: Relation between the kinetic and hydrodynamic descriptions. *Zh. Eksper. Teor. Fiz.*, 1994, **106**, 161–171.
8. DINARIYEV, O. Yu., Kinetic theory of a relativistic gas with internal degrees of freedom in the presence of weak sources. *Zh. Eksper. Teor. Fiz.*, 1995, **107**, 6, 1877–1894.
9. DINARIYEV, O. Yu., Transition from a kinetic to a non-local hydrodynamic description for a non-relativistic gas. *Izv. Vuzov. Fizika*, 1995, 2, 95–99.
10. DINARIYEV, O. Yu., Propagation of weak waves in a gas: The transition from a kinetic to a hydrodynamic description. *Akust. Zh.*, 1995, **41**, 415–420.
11. DINARIYEV, O. Yu., Non-local hydrodynamics of a quantum relativistic many-particle gas. *Zh. Eksper. Teor. Fiz.*, 1995, **107**, 1573–1586.
12. DINARIYEV, O. Yu., Properties of hydrodynamic kernels of quantum field models. *Izv. Vuzov Fizika*, 1995, 7, 34–38.
13. DINARIYEV, O. Yu., Non-local hydrodynamics in the quantum field model φ^4. *Teor. Mat. Fiz.*, 1996, **108**, 1, 50–68.
14. DINARIYEV, O. Yu., On the relation between classical transport coefficients and hydrodynamic kernels for quantum field models. *Izv. Vuzov. Fizika*, 1996, 8, 61–66.
15. DINARIYEV, O. Yu., Non-local hydrodynamics of a quantum many-particle system at zero temperature. *Izv. Vuzov Fizika*, 1997, 8, 37–42.
16. DINARIYEV, O. Yu., On the non-negativity of entropy production in the hydrodynamics of a quantum many-particle system. *Izv. Vuzov Fizika*, 1998, 5, 60–66.

17. DINARIYEV, O. Yu., Equivalence of classical statistical mechanics and non-local hydrodynamics in a certain class of external forces. *Izv. Vuzov Fizika*, 1998, 3, 23–28.
18. RUDYAK, V. Ya., *Statistical Theory of Dissipative Processes in Gases and Liquids*. Nauka, Novosibirsk, 1987.
19. RUDYAK, V. Ya., Non-local solution of the Boltzmann equation. *Zh. Tekh. Fiz.*, 1995, **65**, 11, 29–40.
20. ZUBAREV, D. N., *Non-equilibrium Statistical Thermodynamics*. Nauka, Moscow, 1971.
21. COLEMAN, B. D., Thermodynamics of materials with memory. *Arch. Ration. Mech. and Analysis*, 1964, **17**, 1–46.
22. DAY, W. A., *The Thermodynamics of Simple Materials with Fading Memory*. Springer, Berlin, 1972.
23. TRUESDELL, C., *Introduction à la mécanique rationelle des milieux continues*. Masson, Paris, 1974, **16**, 367.
24. SOBOLEV, S. L., Local-non-equilibrium models of transport processes. *Uspekhi Fiz. Nauk*, 1977, **167**, 1095–1106.
25. DINARIYEV, O. Yu., On certain properties of relaxation kernels in systems with heredity. *Dokl. Akad. Nauk SSSR*, 1989, **309**, 615–618.
26. DINARIYEV, O. Yu., On material relations for a fluid with heredity and non-locality. *Dokl. Akad. Nauk SSSR*, 1991, **316**, 67–71.
27. DINARIYEV, O. Yu., Material relations for a relativistic fluid with non-locality. *Dokl. Akad. Nauk SSSR*, 1991, **319**, 356–359.
28. DIEUDONNÉ, J., *Foundations of modern analysis*. Academic Press, New York, 1960, XIV, 361.
29. STEIN, E. M. and WEISS, G., *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, NJ, 1971.
30. STRATONOVICH, R. L., *Non-equilibrium Non-linear Thermodynamics*. Nauka, Moscow, 1985.
31. CATTANEO, C., Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée. *C. r. Acad. sci*, 1958, **247**, 4, 431–433.
32. DINARIYEV, O. Yu., The velocity of propagation of waves for transport processes with relaxation. *Dokl. Akad. Nauk SSSR*, 1988, **301**, 5, 1095–1097.
33. DINARIYEV, O. Yu., The velocity of signal propagation in a fluid with relaxation. *Prikl. Mat. Mekh.*, 1990, **54**, 1, 59–64.
34. DINARIYEV, O. Yu., Signal propagation in a relativistic fluid with viscosity and heat conduction. *Prikl. Mat. Mekh.*, 1992, **56**, 2, 250–259.
35. DINARIYEV, O. Yu., Signal propagation for transport processes with a space-time non-locality *Dokl. Ross. Akad. Nauk*, 1992, **327**, 4–6, 481–484.
36. VLADIMIROV, V. S. and SERGEYEV, A. G., Complex Analysis in a Tube of the Future. In *Advances in Science and Technology. Mathematical Problems. Basic Trends*. Vsesoyuz. Inst. Nauch. i Tekh. Inform., 1985, 8, 191–266.

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